

## A Caveat on Building Nonlocal Models of Cosmology

N. C. Tsamis<sup>1\*</sup> and R. P. Woodard<sup>2†</sup>

<sup>1</sup> *Institute of Theoretical Physics & Computational Physics,  
Department of Physics, University of Crete,  
GR-710 03 Heraklion, GREECE*

<sup>2</sup> *Department of Physics, University of Florida,  
Gainesville, FL 32611, UNITED STATES*

### ABSTRACT

Nonlocal models of cosmology might derive from graviton loop corrections to the effective field equations from the epoch of primordial inflation. Although the Schwinger-Keldysh formalism would automatically produce causal and conserved effective field equations, the models so far proposed have been purely phenomenological. Two techniques have been employed to generate causal and conserved field equations: either varying an invariant nonlocal effective action and then enforcing causality by the ad hoc replacement of any advanced Green's function with its retarded counterpart, or else introducing causal nonlocality into a general ansatz for the field equations and then enforcing conservation. We point out here that the two techniques access very different classes of models, and that neither one of them may represent what would actually arise from fundamental theory.

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\* e-mail: tsamis@physics.uoc.gr

† e-mail: woodard@phys.ufl.edu

# 1 Introduction

Although quantum corrections to the effective field equations are known to be nonlocal, this is not typically a macroscopic effect for two reasons:

1. The virtual particles circulating in the loops are massive; and
2. When massless virtual particles do participate, they either lack self-interactions or else these self-interactions have a high dimension which renders them ineffective on large scales.

The nonlocal effects of perturbative quantum field theory derive from inverse differential operators, and those of massive particles can be expanded in a series of local, higher derivatives,

$$\frac{i}{\partial^2 - m^2} = -\frac{i}{m^2} \left[ 1 + \frac{\partial^2}{m^2} + \frac{\partial^4}{m^4} + \dots \right]. \quad (1)$$

The massless photons of quantum electrodynamics (QED) do produce observable macroscopic effects [1], but these reduce to slight rescalings of “hard” (i.e., without corrections from low virtual momenta) rates and cross sections because photons lack self-interactions [2]. The same result is true for gravitons with zero cosmological constant because their self-interactions are of dimension five ( $\hbar\partial\hbar\partial\hbar$ ) [2]. Long-range graviton corrections to the Newton [3, 4] and Coulomb [5] potentials are suppressed for the same reason,

$$\Phi(r) = \Phi_{\text{classical}} \left\{ 1 + \frac{\#G}{r^2} + \dots \right\}. \quad (2)$$

( $G$  is Newton’s constant.) Macroscopic effects from (nearly) massless neutrinos are even smaller because the weak interaction has dimension six [6, 7].

Quantum chromodynamics (QCD) consists, in perturbation theory, of quarks and massless gluons with self-interactions of dimension four. In this case infrared effects are so strong that the macroscopic spectrum of hadrons and baryons is entirely different from the quarks and gluons of perturbation theory. One could presumably follow the causal evolution of this transformation — at least its opening stages — by releasing a prepared state of free QCD vacuum and employing the Schwinger-Keldysh formalism [8, 9, 10, 11, 12, 13, 14, 15]. Precisely this sort of evolution must have occurred during the early universe around the time of the QCD phase transition.

When a nonzero cosmological constant is included, quantum gravity consists of massless gravitons with a self-interaction of *dimension three*. It ought therefore to suffer even stronger infrared effects than QCD. Veneziano showed that the infrared effects of a massless scalar with a dimension three coupling are so strong that the Bloch-Nordsieck procedure fails to produce finite rates and cross sections [16]. What happens instead is that the vacuum decays, and one can indeed follow the first stages of this process using the Schwinger-Keldysh formalism [17].

Polyakov early suggested that infrared effects in quantum gravity with a positive cosmological constant  $\Lambda$  should be so strong that they screen the bare  $\Lambda$  [18]. Of course explicit computation of anything in quantum gravity is very difficult, the more so in this case because one requires the ultra-violet finite part and because the appropriate background geometry is de Sitter rather than flat space. However, what computations have been made [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] do include a number of effects [19, 20, 22, 23, 24, 27, 28, 29, 30, 31] that show secular corrections which grow like  $G\Lambda \ln[a(t)]$ , where  $a(t)$  is the scale factor [32, 33]. Over the course of a prolonged phase of de Sitter expansion the factor of  $\ln[a(t)]$  must eventually overwhelm the small loop-counting parameter of  $G\Lambda$ , at which point perturbation theory breaks down.

These considerations have prompted a proposal for simultaneously resolving the problem of the cosmological constant and providing a natural model of primordial inflation [34, 35]. The proposal is based on three contentions:

- That the bare cosmological constant is not unreasonably small;
- That this triggered primordial inflation; and
- That inflation was brought to an end by the gradual accumulation of self-gravitation between the infrared virtual gravitons which are ripped out of the vacuum by inflation.

The third item on this list is controversial [36, 37], however, there seems little doubt that inflation does create an ensemble of infrared gravitons [38]. This is what caused the primordial tensor spectrum [39] which the BICEP2 detector has recently claimed to resolve [40]. It is difficult to understand why these gravitons would not attract one another, at least a little, or how the effect, which starts from zero, can avoid growing stronger as more of the newly created gravitons come into causal contact with one another. Indeed,

one can easily show that only the time lag imposed by causality prevents an inflating universe from experiencing gravitational collapse after fewer than ten e-foldings [35]!

As plausible as quantum gravitational back-reaction might seem, there is no simple way of analytically following the process by which it might stop inflation, and hence no simple way of making testable predictions. Just accessing the initial stages of the process requires a 2-loop computation, which represents a year's effort [19], and will only show fractional corrections to the expansion rate of the form  $-(G\Lambda)^2 \ln[a(t)]$  [35]. The full series of leading infrared logarithms takes the form [32, 33],

$$H(t) = \sqrt{\frac{1}{3}\Lambda} \left\{ 1 - G\Lambda \sum_{\ell=2}^{\infty} \left( G\Lambda \ln \left[ \frac{a(t)}{a(t_i)} \right] \right)^{\ell-1} \right\}, \quad (3)$$

where  $t_i$  is the beginning of inflation. From this series we see that perturbation theory breaks down when  $\ln[a(t)/a(t_i)] \sim 1/G\Lambda$ . Evolving beyond this point would require a nonperturbative resummation technique. There seems to be some chance that such a technique can be devised because Starobinsky and Yokoyama were able to find one for scalar potential models which also exhibit infrared logarithms [41, 42, 32]. Their technique has been generalized to scalars which interact with photons [33], and to scalars which interact with fermions [43], but the generalization to quantum gravity has not yet been accomplished [27, 44, 45].

In the absence of a nonperturbative resummation technique one approach has been to explore simple ansätze for the most cosmologically significant part of the effective field equations [46]. These effective field equations must be nonlocal in order to recover known secular dependence of the perturbative result (3) because the de Sitter limit of any local curvature degenerates to factors of  $\Lambda$  times sums of products of the metric. In addition to attempting to represent the quantum gravitational back-reaction to primordial inflation [47, 48, 49], nonlocal modifications of gravity have also been invoked to describe the current phase of cosmic acceleration without recourse to dark energy [50, 51, 52, 53, 54, 55, 56], to provide a metric-based realization [57, 58, 59, 60, 61, 62, 63, 64] of Milgrom's MOdified Newtonian Dynamics (MOND) [65, 66, 67], and to solve a variety of other problems [68].

One problem with nonlocal modifications of gravity is generating field equations which are both causal and conserved. If the modification was derived from fundamental theory these two requirements would follow nat-

urally from the Schwinger-Keldysh formalism. However, because no such derivation is currently possible, two entirely phenomenological approaches have been followed instead:

1. Proceeding from a general causal ansatz for the field equations and then determining free functions to enforce conservation [47, 48, 49]; or
2. Varying an invariant action whose nonlocality derives from inverse differential operators — which ensures conservation — and then enforcing causality by replacing all advanced Green’s functions with their retarded cousins [57, 50, 53, 56, 64].

The point of this work is to demonstrate that the two approaches do not access the same range of models, and that neither may include the sort of models which would actually come from the Schwinger-Keldysh formalism.

This paper has five sections of which the first is coming to an end. In section 2 we present general models of the two types, assuming their nonlocality derives from the inverse scalar d’Alembertian acting on the Ricci scalar. In section 3 we demonstrate that no choice of the various free functions will make the models agree for more than a single expansion history. Section 4 contrasts the nonlocality of these simple models with what one actually gets from perturbative corrections in the Schwinger-Keldysh formalism. Our conclusions comprise section 5.

## 2 Two Classes of Models

The point of this section is to present general representatives from the two classes of models described above in section 1. We make the additional requirement that their nonlocality derives from acting on the Ricci scalar with the inverse of the scalar covariant d’Alembertian,

$$\square \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right) . \quad (4)$$

We define the inverse of  $\square$  with retarded boundary conditions. In each case we give the model’s correction  $\Delta G_{\mu\nu}$  to the classical Einstein tensor, so that the effective field equations of pure gravity take the form,

$$G_{\mu\nu} + \Delta G_{\mu\nu}[g] = -\Lambda g_{\mu\nu} . \quad (5)$$

We also specialize each class of models to the homogeneous, isotropic and spatially flat geometry relevant to cosmology,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x} \quad \Longrightarrow \quad H(t) \equiv \frac{\dot{a}}{a} . \quad (6)$$

## 2.1 Perfect Fluid-Based Models

The first class of models is based on assuming that the most cosmologically relevant part of the quantum gravitational stress tensor takes the perfect fluid form [47, 48, 49],

$$T_{\mu\nu}[g] = \left( \rho[g] + p[g] \right) u_\mu[g] u_\nu[g] + p[g] g_{\mu\nu} \quad , \quad g^{\mu\nu} u_\mu[g] u_\nu[g] = -1 . \quad (7)$$

Of course this makes the correction to the Einstein tensor,

$$\Delta G_{\mu\nu}[g] = -8\pi G T_{\mu\nu}[g] . \quad (8)$$

We take the pressure to be a general function of  $\frac{1}{\Box}R$ , consistent with the series (3) of perturbative leading logarithms,

$$p[g] = \Lambda^2 f\left(-G\Lambda \frac{1}{\Box}R\right) . \quad (9)$$

The energy density  $\rho[g]$  and 4-velocity  $u_\mu[g]$  are then determined by conservation. It can be shown [47] that all models of this type experience a long period of inflation, followed by a phase of oscillations, provided only that the function  $f(Z)$  grows monotonically and without bound. If matter couplings are added, which permit the dissipation of energy, it becomes plausible that the oscillatory epoch ends in a normal epoch of radiation domination. At this point  $R = 0$ , and  $\frac{1}{\Box}R$  becomes constant, so  $8\pi G T_{\mu\nu} = +\Lambda g_{\mu\nu}$  [48].

In the cosmological geometry (6) the only nonzero components of the affine connection are,

$$\Gamma^i_{j0} = H\delta^i_j \quad , \quad \Gamma^0_{ij} = Hg_{ij} . \quad (10)$$

The nonzero components of the Riemann tensor are,

$$R^0_{i0j} = (\dot{H} + H^2)g_{ij} \quad , \quad R^i_{jkl} = H^2 \left( \delta^i_k g_{j\ell} - \delta^i_\ell g_{jk} \right) . \quad (11)$$

Tracing produces the nonzero components of the Ricci tensor, and tracing again gives the Ricci scalar,

$$R_{00} = -3(\dot{H} + H^2) \quad , \quad R_{ij} = (\dot{H} + 3H^2)g_{ij} \quad , \quad R = 6\dot{H} + 12H^2 . \quad (12)$$

It follows that the two nonzero components of the Einstein tensor are,

$$G_{00} = 3H^2 \quad , \quad G_{ij} = -(2\dot{H} + 3H^2)g_{ij} . \quad (13)$$

When acted on an arbitrary function of time  $F(t)$ , the inverse scalar d'Alembertian takes the form,

$$\frac{1}{\square}F = - \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') F(t'') . \quad (14)$$

In the cosmological geometry (6) the pressure is,

$$p(t) = \Lambda^2 f \left( G\Lambda \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') \left[ 6\dot{H}(t'') + 12H^2(t'') \right] \right) . \quad (15)$$

Enforcing conservation implies the 4-velocity field and energy density are,

$$u_\mu = -\delta_\mu^0 \quad , \quad \rho(t) = -p(t) + \frac{1}{a^3(t)} \int_{t_i}^t dt' a^3(t') \dot{p}(t') . \quad (16)$$

Hence the nonzero components of  $\Delta G_{\mu\nu}$  are,

$$\Delta G_{00} = -8\pi G\rho \quad , \quad \Delta G_{ij} = -8\pi G p g_{ij} . \quad (17)$$

## 2.2 Action-Based Models

The second class of models consists of a Lagrangian which is a general function of  $\frac{1}{\square}R$ ,

$$\Delta\mathcal{L} = \Lambda^2 h \left( -G\Lambda \frac{1}{\square}R \right) \sqrt{-g} . \quad (18)$$

Varying the integral of (18) produces conserved but acausal field equations. Causality is enforced, without disturbing conservation, by the ad hoc replacement of every advanced Green's function with its retarded cousin [57, 50, 53, 56, 64],

$$\Delta G_{\mu\nu}[g] \equiv \frac{16\pi G}{\sqrt{-g}} \left( \frac{\delta\Delta S[g]}{\delta g^{\mu\nu}} \right)_{\text{adv} \rightarrow \text{ret}} . \quad (19)$$

This works because conservation depends only on the relation  $\square \cdot \frac{1}{\square} = 1$ , which is obeyed by both advanced and retarded Green's functions.

A quicker way to derive  $\Delta G_{\mu\nu}[g]$  is to follow Nojiri and Odintsov [69] in localizing the theory through the introduction of auxiliary scalar fields. Our model (18) requires two scalars:  $\phi$  to stand for  $\frac{1}{\square}R$ , and a Lagrange multiplier  $\xi$  whose variation implies  $\square\phi = R$ . The localized Lagrangian density is,

$$\Delta\mathcal{L} \longrightarrow \Lambda^2 h(-G\Lambda\phi) \sqrt{-g} - \left[ \partial_\mu \xi \partial_\nu \phi g^{\mu\nu} + \xi R \right] \sqrt{-g}. \quad (20)$$

The scalar field equations are,

$$\frac{1}{\sqrt{-g}} \frac{\delta \Delta S}{\delta \xi} = \square\phi - R = 0, \quad (21)$$

$$\frac{1}{\sqrt{-g}} \frac{\delta \Delta S}{\delta \phi} = \square\xi - G\Lambda^3 h'(-G\Lambda\phi) = 0. \quad (22)$$

The correction to the Einstein tensor is,

$$\Delta G_{\mu\nu} = 8\pi G \left\{ \left[ -\Lambda^2 h(-G\Lambda\phi) + \partial_\rho \xi \partial_\sigma \phi g^{\rho\sigma} \right] g_{\mu\nu} - 2\partial_{(\mu} \xi \partial_{\nu)} \phi - 2 \left[ G_{\mu\nu} + g_{\mu\nu} \square - D_\mu D_\nu \right] \xi \right\}, \quad (23)$$

where  $D_\mu$  stands for the covariant derivative and parenthesized indices are symmetrized.

It is clear from (20) that one linear combination of the auxiliary scalars would be a ghost if they were truly independent dynamical variables [53, 56]. To avoid this we follow the same procedure that was successfully invoked to avoid ghosts in another nonlocal cosmology model which is based on  $\frac{1}{\square}R$  [53, 56]. The trick is just to define each scalar with vanishing initial value data so that we can express the solutions to equations (21-22) in terms of the retarded Green's function,

$$\phi[g] \equiv \frac{1}{\square}R, \quad \xi[g] \equiv \frac{G\Lambda^3}{\square} h'(-G\Lambda\phi[g]). \quad (24)$$

Substituting (24) in (23) gives the same causal, conserved and ghost-free effective field equations that follow from the partial integration trick (19).



It remains to specialize relations (23) and (24) to the geometry (6) of cosmology. The auxiliary scalars are,

$$\phi(t) = - \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') \left[ \dot{H}(t'') + 12H^2(t'') \right], \quad (25)$$

$$\xi(t) = -G\Lambda^3 \int_{t_i}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') h'(-G\Lambda\phi(t'')). \quad (26)$$

The nontrivial components of  $\Delta G_{\mu\nu}$  are,

$$\Delta G_{00} = 8\pi G \left[ \Lambda^2 h - \dot{\xi}\dot{\phi} - 6H\dot{\xi} - 6H^2\xi \right], \quad (27)$$

$$\Delta G_{ij} = -8\pi G \left[ \Lambda^2 h + 2G\Lambda^3 h' + \dot{\phi}\dot{\xi} + 2H\dot{\xi} - (4\dot{H} + 6H^2)\xi \right] g_{ij}. \quad (28)$$

### 3 Why and How They Disagree

Each of the models described in section 2 depends upon an arbitrary function of the nonlocal quantity  $Z \equiv -G\Lambda \frac{1}{\square} R$ . If these models represent the same physics then it must be possible to define the function  $f(Z)$  of the perfect fluid model in terms of the function  $h(Z)$  of its action-based analog so that both models give the same  $\Delta G_{\mu\nu}$ . We would need this agreement to hold not only for a general cosmological geometry (6) but also for perturbations around this geometry. However, it isn't even possible to make the two classes of models agree for a general expansion history  $a(t)$ . To see this, note that getting the same pressure from expressions (17) and (28) requires,

$$\begin{aligned} f(-G\Lambda\phi(t)) &= h(-G\Lambda\phi(t)) + 2G\Lambda h'(-G\Lambda\phi(t)) \\ &\quad + \frac{1}{\Lambda^2} \left\{ \dot{\phi}(t)\dot{\xi}(t) + 2H(t)\dot{\xi}(t) - [4\dot{H}(t) + 6H^2(t)]\xi(t) \right\}. \end{aligned} \quad (29)$$

The terms on the first line are no problem but those on the second line preclude general agreement between the two models because they depend upon the Hubble parameter and its derivatives, as well as on derivatives and integrals of what should be the single independent variable  $Z(t) = -G\Lambda\phi(t)$ . We can choose the relation between  $f(Z)$  and  $h(Z)$  to make the models coincide for one particular expansion history  $a(t)$  but they will not agree for other expansion histories.

To see the problem in more detail, let us work out the relation between  $f(Z)$  and  $h(Z)$  which is needed to make the models agree for the de Sitter expansion history  $a(t) = e^{H_i t}$ , where  $3H_i^2 = \Lambda$ . Evaluating expression (25) for de Sitter reveals that it is an excellent approximation, after a long period of inflation, to regard  $\phi(t)$  as a linear function of time,

$$\phi(t) = -4H_i(t-t_i) + \frac{4}{3} - \frac{4}{3}e^{-3H_i(t-t_i)} \approx -4H_i\Delta t. \quad (30)$$

Substituting this approximation into expression (26) and assuming that the  $t''$  integration is dominated by the growth of  $a^3(t'')$  results in an equally valid approximation for  $\xi(t)$ ,

$$\xi(t) \approx -\frac{1}{4}\Lambda h\left(-G\Lambda\phi(t)\right). \quad (31)$$

Similar approximations for the derivatives are,

$$\dot{\phi}(t) \approx -4H_i, \quad \dot{\xi}(t) \approx -G\Lambda^2 H_i h'\left(-4G\Lambda\phi(t)\right). \quad (32)$$

Substituting relations (30), (31) and (32) in (29) implies that the two models will agree for de Sitter provided,

$$f(Z) \approx \frac{3}{2}h(Z) + \frac{8}{3}G\Lambda h'(Z). \quad (33)$$

Suppose relation (33) pertains, which will make the two models agree during de Sitter inflation. Now consider an expansion history with a long period of nearly de Sitter inflation, followed by an intermediate phase in which the Ricci scalar becomes negative, and then a long period of perfect radiation domination with  $R(t) = 0$ ,

$$t_i < t < t_1 \implies \text{Inflation with } R(t) > 0, \quad (34)$$

$$t_1 < t < t_2 \implies \text{Oscillation with } R(t) < 0, \quad (35)$$

$$t_2 < t < t_3 \implies \text{Radiation with } R(t) = 0. \quad (36)$$

This is an important expansion history because the perfect fluid model (with matter) follows it for any function  $f(Z)$  which increases monotonically and without bound [47]. During the phase of radiation domination we can write,

$$t_2 < t < t_3 \implies \phi(t) = -\int_{t_i}^{t_2} \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') R(t'') - \int_{t_2}^t \frac{dt'}{a^3(t')} \int_{t_i}^{t_2} dt'' a^3(t'') R(t''). \quad (37)$$

We can obviously choose the expansion history  $a(t)$  to make the rightmost integral of (37) vanish because this represents only a single condition on the infinite number of points at which  $a(t'')$  can be specified for  $t_i < t'' < t_2$ . Suppose this has been done, which makes  $\phi(t)$  constant and its first derivative zero,

$$t_2 < t < t_3 \implies \phi(t) = -\int_{t_i}^{t_2} \frac{dt'}{a^3(t')} \int_{t_i}^{t'} dt'' a^3(t'') R(t'') \equiv \phi_{\text{cr}} \quad , \quad \dot{\phi}(t) = 0 . \quad (38)$$

This means that the perfect fluid pressure becomes exactly constant during radiation domination,

$$t_2 < t < t_3 \implies p(t) = \Lambda^2 f(-G\Lambda\phi_{\text{cr}}) . \quad (39)$$

It is easy to see that the pressure of the action-based model during the epoch of radiation domination differs from (39). Note first that, if  $H(t_2) \equiv H_2$  and  $a(t_2) \equiv a_2$ , then we have,

$$t_2 < t < t_3 \implies H(t) = \frac{H_2}{1+2H_2(t-t_2)} \quad , \quad a(t) = a_2 \left[ 1+2H_2(t-t_2) \right]^{\frac{1}{2}} . \quad (40)$$

Substituting expressions (38) and (40) into (26) gives the following late time form for  $\xi(t)$ ,

$$t_2 < t < t_3 \implies \xi(t) = -G\Lambda^3 h'(-G\Lambda\phi_{\text{cr}}) \times \frac{t^2}{5} + O(t) . \quad (41)$$

We infer from relation (29) that agreement between the two models for  $t_2 < t < t_3$  requires,

$$f(-G\Lambda\phi_{\text{cr}}) = h(-G\Lambda\phi_{\text{cr}}) + \frac{3}{2}G\Lambda h'(-G\Lambda\phi_{\text{cr}}) . \quad (42)$$

Subtracting (42) from (33) gives,

$$\frac{1}{2}h(-G\Lambda\phi_{\text{cr}}) + \frac{7}{6}G\Lambda h'(-G\Lambda\phi_{\text{cr}}) = 0 , \quad (43)$$

which is not obeyed for a general function  $h(Z)$ . Many, many similar disagreements can be derived.

## 4 Schwinger-Keldysh Field Equations

No one knows what form the Schwinger-Keldysh effective field equations of quantum gravity might take beyond perturbation theory and for an arbitrary cosmological geometry (6). However, quite a bit of experience has been gained working at one and two loop orders on de Sitter background [34, 21, 23, 26, 27, 30, 31, 70, 71]; that is, with  $a = e^{H_i t}$  in the cosmological background (6). The point of this section is to discuss three potentially significant differences between the nonlocality manifest in these equations and the  $1/\square$  nonlocality so far explored in phenomenological models:

1. The effective field equations typically involve nonlinear powers of inverse differential operators;
2. One consequence of this nonlinearity is that the effective field equations typically involve the real part of the propagator as well as the retarded Green's function; and
3. The effective field equations typically involve inverse tensor differential operators in addition to the inverse scalar d'Alembertian.

To simplify the discussion of the first two issues we will suppress the indices of the graviton field, its propagator, and its interaction vertices:  $h_{\mu\nu}(x) \rightarrow h(x)$ . In this notation the effective field equations take the form,

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int d^4 x_2 h(x_2) \dots \int d^4 x_n h(x_n) \Gamma^{(n)}(x, x_2, \dots, x_n) = 0, \quad (44)$$

where  $\Gamma^{(n)}(x_1, x_2, \dots, x_n)$  is the full one-particle-irreducible (1PI)  $n$ -point function, including the classical contributions. Figure 1 shows two of the many diagrams which contribute to the 1PI graviton 1-point function at two loop order [19]. The diagram on the left involves the product of two coincident propagators,

$$\Gamma^{(1a)}(x) = \frac{\kappa^4}{2} \left[ i\Delta(x; x) \right]^2, \quad (45)$$

where  $\kappa^2 \equiv 16\pi G$  is the loop-counting parameter of quantum gravity and  $i\Delta(x; x')$  is the propagator. The diagram on the right of Fig. 1 involves an integral of the product of three propagators,

$$\Gamma^{(1b)}(x) = -\frac{i\kappa^4}{3!} \int d^D x' \left[ i\Delta(x; x') \right]^3. \quad (46)$$



Figure 1: Two of the many two loop diagrams which contribute to the 1PI graviton 1-point function [19]. Wavy lines represent graviton propagators.

Of course propagators are the inverses of differential operators so expressions (45-46) manifest the nonlinearity which was point 1 above. Both diagrams are divergent so they must be evaluated in  $D$  spacetime dimensions, before being combined with the appropriate counterterms to give the  $D \rightarrow 4$  limit which appears in the full 1-point function.

In reality the vertices of quantum gravity contain derivatives and index structures which preclude more than a single one of the propagators in expressions (45-46) from contributing an infrared logarithm. The part of the graviton propagator which potentially contributes an infrared logarithm is the same as the propagator of a massless, minimally coupled scalar [72, 73],

$$i\Delta(x; x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \left(\frac{4}{y}\right)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \left(\frac{4}{y}\right)^{\frac{D}{2}-2} + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \ln(aa') + K \right. \\ \left. + \sum_{n=1}^{\infty} \left[ \frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \left(\frac{y}{4}\right)^n - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-\frac{D}{2}+2} \right] \right\}, \quad (47)$$

where  $K$  is a  $D$ -dependent constant and  $y$  is the de Sitter length function,

$$y(x; x') \equiv aa' \left[ H_i^2 \|\vec{x} - \vec{x}'\|^2 - (|e^{-H_i t} - e^{-H_i t'}| - i\epsilon)^2 \right]. \quad (48)$$

A detailed study [27] of graviton corrections to the propagation of massless fermions (see Fig. 2) concluded that infrared logarithms derive entirely from the 2nd and 3rd terms on the first line of expression (47) ,

$$\frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \left(\frac{4}{y}\right)^{\frac{D}{2}-2} + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \ln(aa'). \quad (49)$$

The second term of (49) is responsible for the famous secular dependence of the coincidence limit [74, 75, 76] which occurs in the left hand diagrams of



Figure 2: The two primitive one loop diagrams which contribute to the 1PI fermion 2-point function [22]. Wavy lines represent graviton propagators while solid lines with arrows stand for the fermion propagator.

Figures 1 and 2. The first term of (49) vanishes in the dimensionally regulated coincidence limit, but it can and does contribute an infrared logarithm in the diagram to the right of Fig. 2. One of the reasons it has been so difficult to sum the series of leading infrared logarithms is that multiplicative constants of order one derive from *all* parts of the various propagators which do not contribute infrared logarithms, so they cannot be simplified.

One might worry that the infrared logarithms from coincident propagators such as those of expression (45) do not seem to be associated with  $\frac{1}{\square}$  acting on anything. However, Dolgov and Pelliccia have shown — and for a general metric, not just de Sitter — that  $i\Delta(x; x)$  is proportional to  $\frac{1}{\square}$  acting on the trace of the free scalar stress tensor [77]. Of course the relation between the graviton propagator and the scalar propagator is only valid in the cosmological geometry (6) [78] but this is precisely the class of geometries of interest. It is not known what the trace of the scalar stress tensor should be for a general cosmological geometry. This quantity has the same dimension as the product of two curvatures, and the models of section 2 were based on assuming it is  $-\Lambda \times R$ . A later class of models [48] explored the possibility that it is  $R_{00} \times R$ . We shall argue elsewhere that the most interesting part of the trace cannot be the product of any local curvatures [79].

As noted previously, the phenomenological models so far explored employ ad hoc techniques to derive casual and conserved field equations. This is not at all what fundamental theory predicts. The effective field equations of fundamental theory derive from a variant of the usual Feynman rules which is known as the Schwinger-Keldysh formalism [8, 9, 10, 11, 12, 13, 14, 15]. We shall omit a general explanation of this technique and merely describe how it affects expression (46). Schwinger-Keldysh propagators have  $\pm$  polarities at each of their two endpoints. The  $++$  polarity corresponds to the usual Feynman propagator whose real and imaginary parts we can write as,

$$i\Delta_{++}(x; x') = R(x; x') + iI(x; x') . \quad (50)$$

Recall that the real part of the propagator depends upon the quantum vacuum state and is generally nonzero throughout spacetime, whereas the imaginary part is the sum of the advanced and retarded Green's functions and is zero for spacelike separation. The other polarity we require is  $+-$ ,<sup>1</sup>

$$i\Delta_{+-}(x; x') = R(x; x') - i\text{sgn}(t-t')I(x; x') . \quad (51)$$

Note that the  $++$  and  $+-$  polarities agree for  $t < t'$ , whereas they are complex conjugates for  $t > t'$ .

With this explanation we can now discuss point 2 above. It turns out that the Schwinger-Keldysh result for expression (46) is [19],

$$\Gamma_{\text{SK}}^{(1b)}(x) = -\frac{i\kappa^4}{3!} \int d^D x' \left\{ \left[ i\Delta_{++}(x; x') \right]^3 - \left[ i\Delta_{+-}(x; x') \right]^3 \right\} , \quad (52)$$

$$= \frac{\kappa^4}{3!} \int d^D x' \theta(t-t') \left\{ 6R^2(x; x')I(x; x') - 2I^3(x; x') \right\} . \quad (53)$$

Only points  $x'^\mu$  on or within the past lightcone of  $x^\mu$  make nonzero contributions to expression (53) because each term of the integrand involves at least one factor of the imaginary part of the Feynman propagator, which vanishes for spacelike separation. When multiplied by  $\theta(t-t')$  this imaginary part becomes the retarded Green's function. The phenomenological models so far studied all derive their nonlocality from the retarded Green's function, but it is evident from expression (53) that *the real part of the propagator can also contribute*. This might be important because the real part of the propagator depends more strongly on infrared gravitons than does the imaginary part.

We come finally to point 3. Ferreira and Maroto have made a pioneering study of nonlocality from inverting higher spin d'Alembertians [54]. They noted that these inverses generically involve exponentially growing modes, however, this can be easily fixed by adding the same nonminimal couplings which nature provides for the quanta in question. In Lorentz gauge, the photon kinetic operator is not the vector d'Alembertian but rather [80, 81],

$$(D_1)_\mu{}^\nu \equiv \square_\mu{}^\nu - R_\mu{}^\nu . \quad (54)$$

Under the assumptions of homogeneity and isotropy (6) vector fields must take the form,  $V_\mu = V_0(t)\delta_\mu^0$ . Hence the inverse  $F_\mu \equiv (\frac{1}{D_1}V)_\mu$  obeys,

$$V_0(t) = -\ddot{F}_0 - 3H\dot{F}_0 - 3\dot{H}F_0 = -\frac{d}{dt} \left[ \frac{1}{a^3} \frac{d}{dt} (a^3 F_0) \right] . \quad (55)$$

---

<sup>1</sup>The  $--$  and  $-+$  polarities are just the complex conjugates of the  $++$  and  $+-$  polarities, respectively.

Equation (55) is straightforward to solve with retarded boundary conditions,

$$\left(\frac{1}{D_1}V\right)_\mu = -\frac{\delta_\mu^0}{a^3(t)}\int_{t_i}^t dt' a^3(t')\int_{t_i}^{t'} dt'' V_0(t'') . \quad (56)$$

The same sort of fix works for higher spins. In de Donder gauge the graviton kinetic operator is not the tensor d'Alembertian but rather [82],

$$(D_2)_{\mu\nu}{}^{\rho\sigma} \equiv \square_{\mu\nu}{}^{\rho\sigma} - 2R_{\mu}{}^{(\rho}{}_{\nu}{}^{\sigma)} + 2R_{(\mu}{}^{(\rho}\delta_{\nu)}{}^{\sigma)} - \frac{4}{3}R\delta_{\mu}{}^{(\rho}\delta_{\nu}{}^{\sigma)} + \frac{1}{3}Rg_{\mu\nu}g^{\rho\sigma} . \quad (57)$$

The same cosmological symmetries of homogeneity and isotropy restrict any tensor to the form,

$$T_{00} = T_{00}(t) \quad , \quad T_{0i} = 0 = T_{i0} \quad , \quad T_{ij} = T(t)g_{ij} . \quad (58)$$

Hence  $F_{\mu\nu} \equiv (\frac{1}{D_2}T)_{\mu\nu}$  obeys,

$$(D_2)_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} = T_{\mu\nu} \quad \implies \quad \begin{cases} -\frac{1}{a^3}\frac{d}{dt}[a^3\dot{F}_{00}] = T_{00} , \\ -\frac{1}{a^3}\frac{d}{dt}[a^3\dot{F}] = T . \end{cases} \quad (59)$$

Of course this is just the scalar d'Alembertian acting on each component [78] and so the retarded inverse can be expressed as,

$$\left(\frac{1}{D_2}T\right)_{\mu\nu} = -\int_{t_i}^t \frac{dt'}{a^3(t')}\int_{t_i}^{t'} dt'' a^3(t'')T_{\mu\nu}(t'') . \quad (60)$$

One might be tempted to conclude that there is no need to distinguish between  $(D_2)_{\mu\nu}{}^{\rho\sigma}$  and the scalar d'Alembertian, and this is probably correct for perfect fluid models in which the effective field equations are posited directly. However, the distinction does matter for action-based models in which the field equations are obtained by variation because the two differential operators do not agree for a general metric.

## 5 Epilogue

As explained in section 1, our aim in nonlocal model-building is to guess the most cosmologically significant part of the effective field equations which might pertain after the self-gravitation between inflationary gravitons has become nonperturbatively strong. We know the perturbative limit (3), and we



suspect the correct model must be simple, so finding a reasonable ansatz for it seems possible. Nor is this exercise doomed to forever remain speculative. The prediction of a compelling cosmology would be a strong indication that the correct model had been found, and knowing its form might well facilitate a derivation from fundamental theory, just as the stochastic formalism of Starobinsky and Yokoyama [41] for scalar potential models was later derived [32, 42].

It is neither possible, nor even particularly desirable, to incorporate all details of the actual effective field equations. However, it is crucial that our nonlocal ansätze be sufficiently general to recover their most cosmologically significant portions. In this regard it is disturbing that the two techniques so far employed to generate models seem to access different cosmologies. Subsection 2.1 discussed a perfect fluid ansatz based on an arbitrary function  $f(-G\Lambda\frac{1}{\square}R)$ , whereas subsection 2.2 described a general class of actions based on a different arbitrary function  $h(-G\Lambda\frac{1}{\square}R)$ . In section 3 we showed that one cannot choose  $f(Z)$  in terms of  $h(Z)$  so as to make the two models agree for more than a single expansion history. One way to understand the difference between the two classes of models is that the localized form (20) of the action-based model includes a conformal coupling  $\xi R\sqrt{-g}$  whose variation induces terms in the effective field equations which cannot be present in the perfect fluid model.

It is also disturbing that there seem to be significant differences between the existing phenomenological models and the known one and two loop contributions to the effective field equations on de Sitter background. In section 4 we considered three properties of the Schwinger-Keldysh effective field equations which are not shared by the models:

- They typically involve nonlinear powers of inverse differential operators;
- They typically involve the real parts of propagators, in addition to the imaginary parts which appear in the retarded Green's function; and
- They typically involve the inverses of tensor differential operators.

Some of these differences may be more apparent than substantive. For example, although expressions (45) and (46) involve two and three propagators, it is known that only one propagator in each case can provide the crucial secular dependence which is represented by  $\frac{1}{\square}$  in the models. The other propagators need to be present, but they might be well approximated, for the purposes

of cosmology, by simple functions of the curvature. And the close relation (60) between  $\frac{1}{\square}$  and its tensor cousin  $\frac{1}{D_2}$  in the cosmological geometry (6) means that the distinction between the two operators only shows up in the form of some extra terms in the variational equations.

The overall conclusion for now is that model-builders should be aware of the potential for problems with certain ansätze. We suspect that the action-based models are more likely to be correct than the perfect fluid ones because the effective field equations of fundamental theory derive from varying the Schwinger-Keldysh effective action, which is closely related to the in-out effective action. One feature we can already see that existing models fail to correctly describe is the curvature-squared source upon which  $\frac{1}{\square}$  acts. The existing models are based upon taking this source to be either  $-\Lambda \times R$  [47] or  $R_{00} \times R$  [48], which agree for de Sitter but not in general. Preliminary analysis shows that the actual source should not even be local [79].

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